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Anomalous diffusion on fractal lattices with site disorder

S Robillard and A-M S Tremblay

Département de physique et Centre de Recherche en Physique du Solide, Université de Sherbrooke, Sherbrooke, Québec J1K 2R1, Canada

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Abstract. Like random walks on Euclidean lattices, random walks on fractal lattices are modified by a waiting time (site) disorder when the first moment of the waiting time distribution diverges. It is shown that for lattices which support recurrent walks (spectral dimension smaller than two) the inverse of the diffusion exponent in the presence of disorder is increased by the difference between the waiting time fractal dimension and the usual fractal dimension. This hyperscaling relation is derived for Sierpinski gaskets in arbitrary dimension with scale-dependent waiting times. This provides qualitative insight into this problem and the exponent relation derived should also hold for statistically self-similar structures such as percolation clusters. For lattices whose usual spectral dimension is larger than two, a mean-field result holds.

1. Introduction

Anomalous diffusion refers to random walks with mean-square displacement $\langle R^2 \rangle$ which scales as $t^{2\nu}$ with $\nu < \frac{1}{2}$ ($\nu = \frac{1}{2}$ for random walks on Euclidean lattices). Anomalous diffusion occurs, for example, on percolation clusters. Such clusters form, within a correlation length, a statistically self-similar hopping network (locally, though, the hopping rate is either 1 or 0 with probability p and $(1-p)$ respectively and this distribution of hopping rate is not self-similar). Anomalous diffusion also occurs on Euclidean networks with random biases (Fisher 1984) or with power law distributions of waiting times or hopping rates (Alexander 1981, Stephen and Kariotis 1982, Machta 1985).

There are physical problems where one is faced with anomalous properties arising both from an underlying percolation network and from a power law distribution of hopping rates. For example, to compute the conductivity of continuum percolation systems, one is led to introduce for certain classes of problems a mapping to the so-called 'swiss-cheese' model which involves a regular percolation network with a power law distribution of conductivities (Halperin *et al* 1985, Kogut and Straley 1979).

In this paper, we investigate the type of anomalous diffusion which results when both the network and waiting times are such that each can independently produce anomalous diffusion. To this end, we calculate the anomalous diffusion exponent for a random walk on a Sierpinski gasket with a power law waiting time distribution. In other words, we consider a Sierpinski gasket where each bond has a constant hopping rate and the waiting times at the vertices are unequal but distributed in a regular self-similar fashion. We find that the anomalous exponent ν_r is simply related to the exponent ν which occurs when there is a single waiting time, to the fractal dimension and to the waiting time fractal dimension introduced by Machta (1985). We give a

simple physical interpretation of our results. The general exponent relations we find for a recurrent walk should apply, for example, to diffusion on percolation clusters with a power law distribution of waiting times or to the corresponding properties of equivalent elastic or electric network problems. Indeed, it is by now well established that simple regular self-similar structures (such as the Sierpinski gasket) can provide qualitative insight into the behaviour of real disordered systems characterised by statistical self-similarity (Gefen *et al* 1981). For example, general relations between anomalous diffusion, fractal dimension and the exponents characterising spectral properties are easy to check explicitly on regular fractals and apply equally well to percolation clusters. We believe that qualitative insight is of the utmost importance for the type of problems discussed here since the understanding of real systems rests on the properties of stable distributions (Feller 1971). Such distributions sometimes have very counter-intuitive properties.

Anomalous diffusion translates into corresponding anomalous properties for problems mathematically equivalent to the random walk, such as resistor-capacitor networks or mass and (isotropic) spring problems.

In § 2 the model is specified in more detail. Section 3 contains a renormalisation group derivation of our result. A discussion of our work and its relation to more recent papers may be found in § 4.

2. The model

We start from the master equation for the random walk

$$\tau_i \partial P_i(t) / \partial t = \sum_j^{2d} W(P_j - P_i) \quad (1)$$

where $P_i(t)$ is the probability for a random walker to be at site i at time t , τ_i is the waiting time on site i and W is the hopping rate. The sum over j runs on the nearest neighbours of every site i . The Laplace transform of (1) is

$$(-\tau_i s + U) \tilde{P}_i(s) - \sum_j^{2d} W \tilde{P}_j(s) = \tau_i P_i(0) \quad (2a)$$

with

$$\tilde{P}_i(s) = \int_0^\infty e^{-st} P_i(t) dt \quad (2b)$$

and $U = 2dW$ with $2d$ the number of nearest neighbours on a d -dimensional Sierpinski gasket.

Figure 1 illustrates the model in two dimensions. The Sierpinski gasket is made up of constant hopping rates distributed on a self-similar network and the waiting times at the vertices are distributed in a self-similar fashion. The waiting times at the smallest scale are unity and they increase by a factor r at each successive level of the gasket.

The fractal dimension \bar{d} of the usual Sierpinski gasket (identical vertices) is well known: $\bar{d} = \ln(d+1)/\ln(2)$. For our modified gasket, it is natural to introduce a new fractal dimension \bar{d}_τ since the vertices are weighted differently. A natural definition for the fractal dimension \bar{d}_τ of the waiting times is $\tau(L) \sim L^{\bar{d}_\tau}$ where $\tau(L)$ is the sum of all waiting times within a *typical* region of linear size L . This quantity will be different from the usual fractal dimension of the gasket if r (see figure 1) is larger than 3.

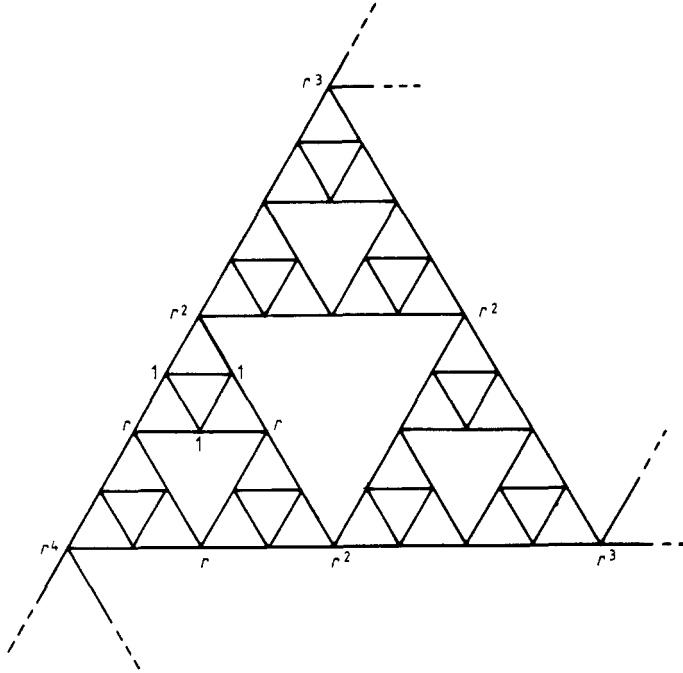


Figure 1. Two-dimensional Sierpinski gasket with self-similar arrangement of site waiting times. At the smallest scale, the waiting time is unity. These times increase by a factor r from one level of the gasket to the next.

To compute the fractal dimension \bar{d}_τ of the waiting time distribution, consider ‘typical’ pieces of size 1, 2, 4, 8 and 16 measured in units of an edge of the smallest triangles. These pieces, labelled (a), (b), (c), (d) and (e), are illustrated in figure 2. The sum of the waiting times in each region is, respectively,

$$\begin{aligned}
 \text{(a)} \quad \tau(1) &= 2 + r \\
 \text{(b)} \quad \tau(2) &= 3 + 2r + r^2 \\
 \text{(c)} \quad \tau(4) &= 9 + 3r + 2r^2 + r^3 \\
 \text{(d)} \quad \tau(8) &= 27 + 9r + 3r^2 + 2r^3 + r^4 \\
 \text{(e)} \quad \tau(16) &= 81 + 27r + 9r^2 + 3r^3 + 2r^4 + r^5.
 \end{aligned}
 \tag{3}$$

For large length scales, $\tau(2^n)$ with n large, we can distinguish two cases: given that the first terms of the polynomials in (3) are powers of $r/3$, we see that

$$\frac{\tau(2^{n+1})}{\tau(2^n)} = 3 \quad \text{for } r < 3 \quad \text{hence } \bar{d}_\tau = \ln(3)/\ln(2) \tag{4a}$$

$$\frac{\tau(2^{n+1})}{\tau(2^n)} = r \quad \text{for } r > 3 \quad \text{hence } \bar{d}_\tau = \ln(r)/\ln(2). \tag{4b}$$

Clearly, our choice of starting point and of successive regions (a), (b), (c), (d), (e) on the figure is not completely general. Most other starting points though give the same result for sufficiently large length scales. The cases which do not satisfy (4a) and (4b) are not very frequent. Analogous considerations give for the d -dimensional Sierpinski

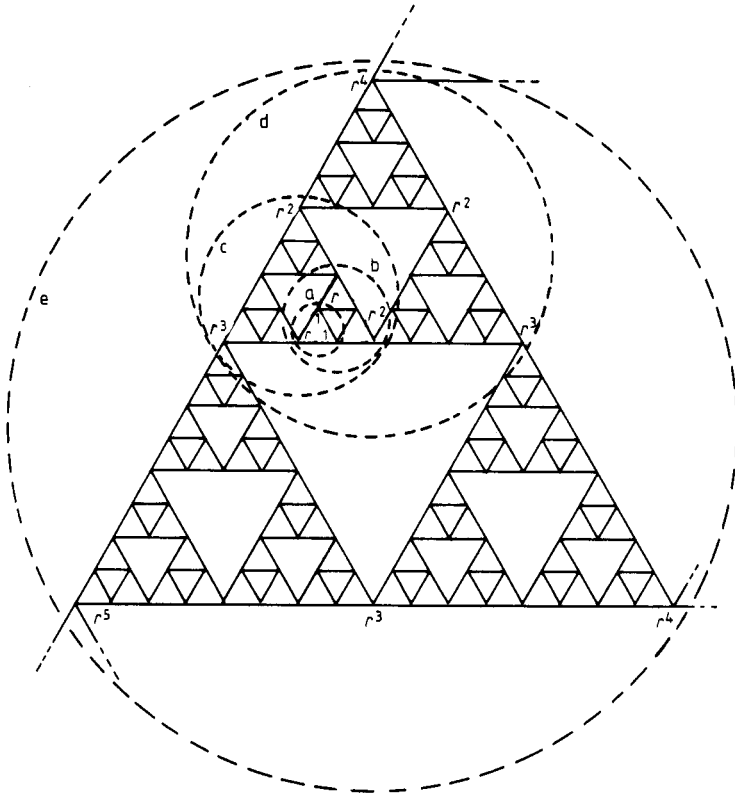


Figure 2. Example of successive length scales considered for the definition of the waiting time fractal dimension \bar{d}_r on Sierpinski gaskets.

gasket

$$\bar{d}_r = \ln(d + 1) / \ln(2) = \bar{d} \quad \text{for } r < d + 1 \tag{5a}$$

$$\bar{d}_r = \ln(r) / \ln(2) \quad \text{for } r > d + 1. \tag{5b}$$

The above geometrical considerations give the regular fractal version of power law waiting time distributions in disordered systems. To make the connection between the fractal dimension \bar{d}_r defined above and the power law characterising the distribution of waiting times in the disordered case, consider the three classes of distributions defined by Alexander *et al* (1981). Let $\mathcal{P}(\tau)$ be the probability density for having a waiting time τ at a given site. Then, with

$$\mathcal{P}(\tau) \sim \tau^{-\alpha-1} \quad \text{for } \tau \rightarrow \infty \tag{6}$$

we have

- class (a): $\alpha > 1$ (first moment finite, ordinary diffusion)
- class (b): $\alpha = 1$ (intermediate case) (7)
- class (c): $0 < \alpha < 1$ (first moment undefined, anomalous diffusion).

Note that $\alpha \rightarrow 1 - \alpha$ in the notation of Alexander *et al* (1981). For the gasket in figure 1, we have by construction three times more waiting times τ than times τr . Hence, in

d dimensions

$$\mathcal{P}(\tau)d(\tau) = (d + 1)\mathcal{P}(r\tau)d(r\tau) \tag{8}$$

which means, by comparison with (6),

$$\alpha = \ln(d + 1)/\ln(r). \tag{9}$$

The value $r = d + 1$ in (5a) and (5b) thus separates cases that correspond exactly to the three classes identified in (7):

$$\begin{aligned} \text{class (a) } & 1 < r < (d + 1) & \bar{d}_\tau &= \ln(d + 1)/\ln(2) = \bar{d} \\ \text{class (b) } & r = d + 1 & \bar{d}_\tau &= \bar{d} \\ \text{class (c) } & r > d + 1 & \bar{d}_\tau &= \ln(r)/\ln(2) = \bar{d}/\alpha. \end{aligned} \tag{10}$$

It is thus convenient to use the waiting time fractal dimension since the different regimes correspond exactly to the different classes. In analogy with the one-dimensional case, we show below that it is only when the waiting time fractal dimension differs from the network fractal dimension (class (c)) that anomalous diffusion due to the waiting time distribution sets in. The marginal class (b) requires special discussion. Our definition of waiting time fractal dimension is analogous to that first introduced by Machta (1985). Note also that the arguments used to define the waiting time fractal dimension involve consideration of ‘typical’ configurations in complete analogy with the arguments one must use to compute sums of variables with distributions of the type of (6) when no formal average exists.

3. Renormalisation group analysis

It is possible, in general, to apply exact position space renormalisation group (PSRG) methods to the Sierpinski gasket. We follow the method proposed by Tremblay and Southern (1983) and Rammal (1984). The generating function \mathcal{F} for this problem is

$$\mathcal{F} = \ln \left(\int \mathcal{D}P \exp[-\frac{1}{2}P'(Ts + H)P] \right) \tag{11}$$

where

$$P' = (P_1, P_2, \dots)$$

$$\int \mathcal{D}P = \prod_i \int dP_i$$

while T is the diagonal matrix containing the waiting times τ_i and H is the matrix defined by writing the left-hand side of (2) as $(Ts + H)P$. Note that the eigenvalues of H are positive; hence the integral in (11) converges.

To generate recursion relations, one performs the integrals in (11) over the variables for the sites at the smallest length scale. The unintegrated variables can be relabelled and the parameters redefined in such a way that the remaining problem is equivalent to the original one but with one fewer level of iteration. The renormalised parameters are, for the two-dimensional gasket,

$$\tau' = r\tau \tag{12a}$$

$$U' = U + 4W^2(\tau s - U)/(\tau s - U - W)(\tau s - U + 2W) \tag{12b}$$

$$W' = -W^2(\tau s - U - 2W)/(\tau s - U - W)(\tau s - U + 2W). \tag{12c}$$

As usual, a ‘constant term’ (i.e. independent of the unintegrated variables) is generated. For a d -dimensional gasket, we use the method of Rammal (1984) to obtain

$$\tau' = r\tau \tag{13a}$$

$$U' = U + 2dW^2[\tau s - U + 2(d-2)W]/[\tau s - U + 2(d-1)W][\tau s - U + (d-3)W] \tag{13b}$$

$$W' = -W^2(\tau s - U - 2W)/[\tau s - U + 2(d-1)W][\tau s - U + (d-3)W]. \tag{13c}$$

One can also take advantage of the arbitrariness in the choice of dummy integration variables to make a non-singular change of variables that reduces the set of three parameters (τ, U, W) to only two (u, v) . To this end, one chooses the variables $Q_i = \sqrt{U}P_i$ at each step of the iteration. In terms of these variables, the parameter U is always equal to unity and the recursion relation for the ‘constant term’ is modified. The new parameters are $u = \tau s/U$ and $v = W/U$. The diagonal terms are $(ur^n - 1)$ where n depends on the position of the site on the Sierpinski gasket. In terms of these new parameters, the recursion relations are

$$u' = \frac{ru[u - 1 + 2(d-1)v][u - 1 + (d-3)v]}{[u - 1 + 2(d-1)v][u - 1 + (d-3)v] + 2dv^2[u - 1 + 2(d-2)v]} \tag{14a}$$

$$v' = -\frac{v^2(u - 1 - 2v)}{[u - 1 + 2(d-1)v][u - 1 + (d-3)v] + 2dv^2[u - 1 + 2(d-2)v]}. \tag{14b}$$

Since we are interested in the asymptotic behaviour $\langle R^2 \rangle \sim t^{2\nu}$ as $t \rightarrow \infty$, we restrict ourselves to small values of the Laplace variable s . The only fixed point of interest is thus $u^* = 0, v^* = 1/2d$ (with the initial value 1 for W). The recursion relations (14) linearised near this fixed point are

$$\begin{aligned} \Delta u' &= \left(\frac{d+3}{d+1}\right) r \Delta u \\ \Delta v' &= \frac{(d+3)}{2(d+1)} \Delta u + (d+3) \Delta v \end{aligned} \tag{15}$$

where $\Delta u \equiv u - u^* = u$ and $\Delta v \equiv v - v^* = v - 1/2d$. The eigenvalues of the transformation are

$$\lambda_1 \equiv b^{y_1} = r[(d+3)/(d+1)] \quad \lambda_2 \equiv b^{y_2} = (d+3). \tag{16}$$

With $b = 2$ as the length rescaling factor, these expressions also define y_1 and y_2 . We proceed to show that the relevant scaling for anomalous diffusion is given by the larger eigenvalue. One first checks explicitly that the ‘constant’ term is non-singular so that the singular part of the generating function per site obeys, after l iterations, the functional equation

$$f^{(l)}(\Delta u^{(l)} r^n - 1, \Delta v^{(l)} + 1/2d) = (b^{\bar{d}})^{-1} f^{(l+1)}(\Delta u^{(l+1)} r^n - 1, \Delta v^{(l+1)} + 1/2d) \tag{17}$$

where the first and second arguments refer respectively to diagonal and off-diagonal terms. Note that the recursion relations are such that the smallest value of n is always zero. With the help of the eigenvectors of (15), the left-hand side of (17) may be rewritten

$$f^{(l)}\left(\lambda_1^l \frac{s}{4} r^n - 1, \frac{1}{2d} + \frac{\lambda_1^l}{2(r-d-1)} \frac{s}{4} - \frac{\lambda_2^l}{2(r-d-1)} \frac{s}{4}\right). \tag{18}$$

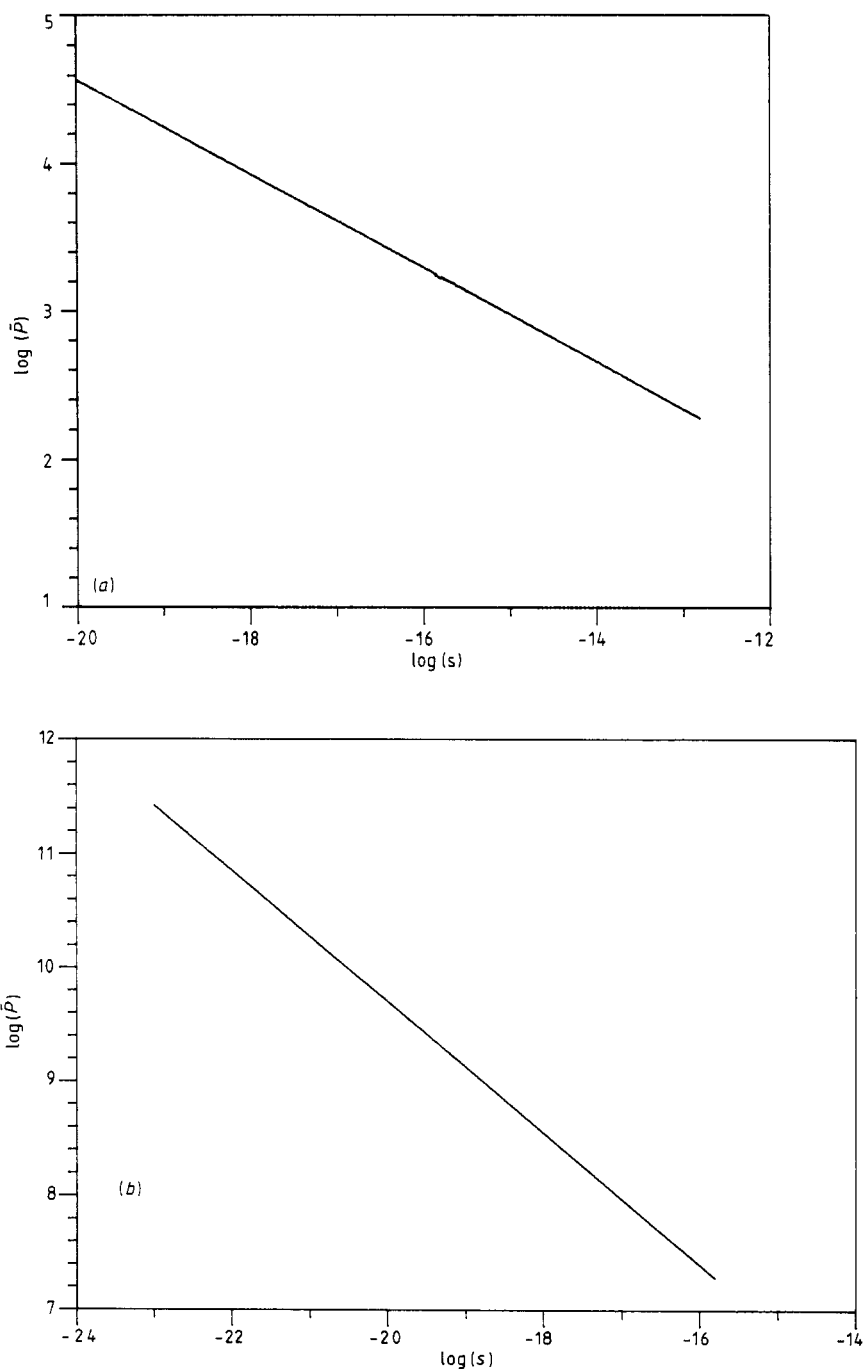


Figure 3. Double logarithmic plot (1200 points) of the Laplace transform of the probability of being on the starting site as a function of the Laplace variable. The slope is $-1 + \bar{d}\nu_r$ and the theoretical values for the two cases considered are $\bar{d} = \ln 3/\ln 2$ and $\nu_r = \ln 2/\ln 5$ in (a) and $\nu_r = \ln 2/\ln(40/3)$ in (b). (a) Waiting time distribution $r = 2$, numerical slope = -0.319 ; analytical slope = -0.317 . (b) Waiting time distribution $r = 8$, numerical slope = -0.575 ; analytical slope = -0.576 .

After a large enough number of iterations, one is sufficiently far away from the singular point to expand the generating function around the point which would have been reached if the smaller of the two eigenvalues had been set equal to zero. Keeping only the leading term, we see that different initial values of s can all be iterated towards the same value S by choosing the number of iterations l such that $S = b^{ly} s$ where, with the definitions (16), $y = y_2$ if $r < (d + 1)$ and $y = y_1$ if $r > (d + 1)$. Since the generating function determines the properties of the random walk, this means that values of s which differ by a ratio $s'/s = b^y$ (corresponding to time ratios $t'/t = 1/b^y$) have related properties at length scales which differ by $R'/R = 1/b$. Since by definition, $\langle R^2 \rangle \sim t^{2\nu}$, we have that $\nu_r = 1/y$. We thus have

$$\text{class (a) } 1 < r < d + 1 \quad \nu_r = \ln(2)/\ln(d + 3) \tag{19a}$$

$$\text{class (c) } r > d + 1 \quad \nu_r = \ln(2)/\ln[r(d + 3)/(d + 1)]. \tag{19b}$$

For class (b), $R = d + 1$ and the two eigenvalues are degenerate and there is no set of eigenvectors that can diagonalise equations (15). This leads to logarithmic corrections. A fuller discussion may be found in Robillard (1985). Note that our analytical results refer only to the ‘envelope’ of the space against time behaviour because the arguments preceding (19) ignore the fact that since b can take only discrete values, the solution to a functional equation such as (17) is not a pure power law but instead is a power law multiplied by a periodic function of $\ln(s)/\ln(\lambda_{\max})$ with period one. (We also use the fact that the eigenvectors are proportional to s (Derrida *et al* 1983).) Similar corrections are encountered in the theory of stable distributions (Di Castro and Jona-Lasinio 1976, Feller 1971, Machta 1985).

To close this section we check our results by numerical calculation of the spectral dimension \tilde{d}_r , defined by the probability $P_{ii}(t)$ for a random walker starting at site i to be at that same site at time t , $P_{ii}(t) \sim t^{-\tilde{d}_r/2}$. This spectral dimension also describes the spectrum of low frequency excitations of an equivalent elastic problem with a distribution of masses but has evidently nothing to do with whether the walk is recurrent or not by contrast with the spectral dimension \tilde{d} which is usually defined for fractals with uniform waiting times (Rammal and Toulouse 1983, Alexander and Orbach 1982). The Laplace transform of the sum over i of $P_{ii}(t)$, which behaves as $s^{-1+\tilde{d}_r/2}$, may be obtained from $\partial f/\partial s \sim s^{-1+\tilde{d}_r/2}$. It is apparent from (17)-(19) that $\tilde{d}_r = 2\tilde{d}\nu_r$. We have checked numerically the value of \tilde{d}_r for the different classes of distributions (6)-(10). Figure 3 illustrates the results. The slope in the various cases is $-1 + \tilde{d}/y = -1 + \tilde{d}\nu_r$ and confirms that y is the larger of the two exponents in (16) and that (20) holds.

4. Discussion and conclusion

Let ν be the exponent characterising anomalous diffusion on a fractal structure, $\langle R^2 \rangle \sim t^{2\nu}$. With a power law distribution of waiting times, diffusion on the same fractal structure is modified. If ν_r denotes the exponent for diffusion in the presence of a waiting time (site) distribution and \tilde{d}_r is the waiting time fractal dimension, then our main result is that

$$(\nu_r)^{-1} - \nu^{-1} = \tilde{d}_r - \tilde{d} \quad (\tilde{d} < 2) \tag{20}$$

with \tilde{d}_r defined by (9) and (10) for the various kinds of distributions of waiting times. As usual (Machta 1985), when an average waiting time exists, $\tilde{d}_r = \tilde{d}$ and the diffusive behaviour is the same as when the waiting time is the same on every site.

The result, (20), has a very simple physical interpretation. From $t \sim R^{1/\nu}$, and from (20)

$$t \sim R^{1/\nu} (R^{\tilde{d}_\tau} / R^{\tilde{d}}). \tag{21}$$

The first factor in (21) is the time necessary to leave a region of size R on the fractal when the waiting time is the same at every site. From the discussion of § 2, $R^{\tilde{d}_\tau}$ is the typical total waiting time in a region of length scale R while $R^{\tilde{d}}$ is the number of sites in that same region. The factor $R^{\tilde{d}_\tau} / R^{\tilde{d}}$ is thus the typical waiting time per site in a region of length scale R . The random walk has thus the same trail (Mandelbrot 1983) (geometrical shape) as the walk without waiting time distribution, but the typical waiting time per site is length scale dependent.

This physical interpretation clearly suggests that our results are far more general than our derivation. We have verified the validity of (20) explicitly on the branching Von Koch curve, for example, but it should also apply to cases where one has only statistical self-similarity as, for example, on percolation clusters with a power law distribution of waiting times. The physics of our result also clearly shows its limitations: the ratio $R^{\tilde{d}_\tau} / R^{\tilde{d}}$ can be interpreted as a renormalised waiting time only if the walk is recurrent, i.e. if the random walk visits all possible sites in a region. Since the trail of the walk (Mandelbrot 1983) is the same as without waiting time disorder, this occurs when the spectral dimension \tilde{d} of the structure with a single waiting time is smaller than 2 (Rammal and Toulouse 1983) or equivalently, when the codimension of recurrence $\tilde{d}/2$ is less than unity (Mandelbrot 1984). This is the case for Sierpinski gaskets in all dimensions, for percolation clusters and for most fractal structures. When the spectral dimension is larger than 2, one should recover the following simple generalisation

$$\nu_\tau = \alpha \nu \quad (\tilde{d} > 2) \tag{22}$$

of the ‘mean-field’ result of Alexander (1981). The mean-field theory for this problem is the continuous-time random walk approximation (CTRW) developed by Sher and Lax (1973) and Sher and Montroll (1975) (Machta 1985). The CTRW problem on fractals has been studied by Blumen *et al* (1984) who derived (22).

For Euclidean lattices, ($\tilde{d} = d, \nu = \frac{1}{2}$) with a power law distribution of waiting times, our result (20) reduces to that of Machta (1985) when one notices that on hypercubic lattices, the spectral dimension is equal to the Euclidean dimension, which means that ‘mean-field’ theory applies above two dimensions. This agreement supports the claim that our result is general. The analysis of Machta also shows clearly that given the properties of stable distributions, even if the waiting time distribution is not a pure power law, the result (20) is obtained.

The following generalisation of heuristic arguments due to Alexander (1981) reproduces both results (20) and (22). Let us consider all walks of N steps which do not include a waiting time larger than τ_{\max} . Such walks will be ‘typical’ if they are very probable, i.e. if $(1 - \int_{\tau_{\max}}^{\infty} \mathcal{P}(\tau) d\tau)^{S_N}$ is close to unity. Here S_N is the average number of distinct sites visited in a walk of N steps. Thus

$$S_N \ll \left(\int_{\tau_{\max}}^{\infty} \mathcal{P}(\tau) d\tau \right)^{-1} \sim \tau_{\max}^\alpha. \tag{23}$$

For these walks, there exists an average waiting time

$$\langle \tau \rangle \sim \int_{\tau_{\max}}^{\infty} \tau \mathcal{P}(\tau) d\tau \sim \tau_{\max}^{1-\alpha} \tag{24}$$

so that one can relate time t and number of steps in the walk through

$$t = N\langle\tau\rangle \sim N\tau_{\max}^{1-\alpha}. \quad (25)$$

The latter relation can be used to find the exponent ν_τ because $N \sim R^{1/\nu}$ with ν the exponent without site disorder, while $t \sim R^{1/\nu_\tau}$. To fix τ_{\max} uniquely, we also require that the mean square fluctuations in the average (24) be small, i.e.

$$S_N \gg \langle\tau^2\rangle/\langle\tau\rangle^2 \sim \tau_{\max}^\alpha. \quad (26)$$

Equations (23) and (26) together can both be satisfied only if

$$S_N \sim \tau_{\max}^\alpha. \quad (27)$$

Equations (25) and (27) yield

$$t \sim NS_N^{(\alpha^{-1}-1)}. \quad (28)$$

Using $N \sim R^{1/\nu}$, $S_N \sim N^{\tilde{d}/2}$ for $\tilde{d} = 2\bar{d}\nu < 2$ and $S_N \sim N$ for $\tilde{d} = 2\bar{d}\nu > 2$ (Rammal and Toulouse 1983) we recover (20) and (22).

To summarise, we have derived the scaling law (20) for anomalous diffusion on a Sierpinski gasket in arbitrary dimensions with waiting times arranged in a self-similar fashion as in figure 1. Our derivation is based on an exact renormalisation group and the results have been checked numerically. The hierarchical arrangement of waiting times allows one to define a waiting time fractal dimension. When the waiting time fractal dimension is equal to the ordinary fractal dimension, the exponents characterising diffusion are the same as when the waiting time is identical on every site. We have also given heuristic arguments which extend our results to the case of non-recurrent lattices (equation (22)). Note that when the waiting times at the sites are statistically independent from the walk itself, some of the results of the heuristic considerations given in (23)–(28) can be made rigorous (Feller 1949, Mandelbrot private communication).

The problem we have considered is interesting in its own right since there is a growing interest in the dynamics of systems with hierarchical arrays of barriers or waiting times (Palmer *et al* 1984, Huberman and Kerszberg 1985, Teitel and Domany 1985, Ogielski and Stein 1985). Teitel and Domany, for example, have stressed the analogy between phase transitions and the change from normal to anomalous diffusion which can occur in one dimension when a parameter characterising a hierarchical array of barriers is allowed to change. This change is analogous to the change between class (a) and class (c) (equation (10)). But we think that perhaps the most relevant aspect of our results for real systems comes from their probable applicability not only to the exactly self-similar problem we have discussed here, but also to statistically self-similar systems (disordered systems, for short) such as percolation clusters with power law distributed waiting times. That scaling laws derived for exactly self-similar objects may also be applicable to statistically self-similar systems is an idea pioneered by Mandelbrot which has proved to be quite useful in many previous contexts. We know of no simple proof in our case that (20) holds in the disordered case. One should beware that the Sierpinski gasket is made mostly of loops while the other fractal we have considered, the Koch curve, has none. Real percolation clusters have both loops and singly connected regions: recent results of Machta *et al* (1985) suggest that in that case, the loops could be irrelevant. However, the model considered by Machta *et al* would map into a random hopping problem instead of the random waiting time problem we have considered here.

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